Connection matrices for ultradiscrete linear problems

2007 J. Phys. A: Math. Theor. 40 12799

(http://iopscience.iop.org/1751-8121/40/42/S22)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 12799–12809 doi:10.1088/1751-8113/40/42/S22

Connection matrices for ultradiscrete linear problems

Chris Ormerod

School of Mathematics and Statistics F07, The University of Sydney, Sydney, Australia

E-mail: chriso@maths.usyd.edu.au

Received 6 December 2006, in final form 7 March 2007
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12799

Abstract
We present theory outlining associated linear problems for ultradiscrete equations. The appropriate domain for these problems is the max-plus semiring. Our main result is that despite the restrictive nature of the max-plus semiring, it is still possible to define a theory of connection matrices analogous to that of Birkhoff and his school for systems of linear difference equations. We use such theory to provide evidence for the integrability of an ultradiscrete difference equation.

PACS numbers: 02.30.Ik, 05.45.−a
Mathematics Subject Classification: 39A13, 33C70, 37J35, 16Y60

The discrete versions of the Painlevé equations [16] can be considered integrable systems. There are many ways in which one may provide evidence for the integrability of difference equations; these include singularity confinement [16, 6] (which is widely regarded as the discrete analogue of the Painlevé property) and algebraic entropy [24]. The approach we wish to extend relies on associated systems of linear difference equations. One may consider a system integrable if the system possesses a Lax pair in the discrete sense [13]. One may also extend this notion so that the derivation of the Lax pair comes as the compatibility condition of a connection matrix preserving deformation [8, 20]. The concept of a connection matrix is one that was first formulated by Birkhoff and his school [1, 2] and was later extended by Ramis and his school [21].

Ultradiscrete equations are the result of applying a limiting process to difference equations [23]. The resulting system is one in which you may restrict the evolution to a discrete set, making them discrete in time, space and state [15]. For this reason, these systems are sometimes called cellular automata [7]. The object of our study will be the ultradiscrete versions of the Painlevé equations [15]. Although one expects this process to preserve integrability, we are still required to provide evidence for the integrability of such equations for them to be called integrable. It was shown that some ultradiscrete integrable systems do possess Lax pairs [9, 17]; we wish to extend this theory by establishing the concept of a connection matrix for these associated linear problems.

This paper aims to provide evidence of integrability for the difference equation

$$\overline{W} + \underline{W} = \max(2W, A + T) - \max(0, A + 2W + T) \tag{1}$$

where $W = W(T)$, $\overline{W} = W(T + Q)$ and $\underline{W} = W(T - Q)$ for some fixed $Q \in \mathbb{R}\backslash\{0\}$. This is an ultradiscretization of a known version of q-P_{III} [9] which we will call u-P_{III}. This ultradiscrete equation was shown to have a Lax pair [9]. We intend to show that this system comes as a compatibility condition of a connection preserving deformation. We do this by introducing some theory that allows us to treat these problems systematically. By utilizing a lift similar to those studied in the context of tropical geometry [18] we are able to explore the properties of these systems on the level of non-Archimedean valuation fields.

For convenience we will introduce the max-plus semiring and their associate linear spaces in section 1. We consider this the natural domain for the ultradiscrete equations. We will introduce the non-Archimedean valuation field we will use to study equations over the max-plus semiring and describe the way in which one lifts a problem to this field. In section 2 we present results pertaining to linear systems of difference equations over the semiring. As an application of this theory we present the derivation of (1) in section 3.

1. The max-plus semiring

Given an additively and topologically closed subset of the real numbers, $U \subset \mathbb{R}$, we construct the semiring, $S = U \cup \{-\infty\}$, by adjoining the binary operations max and +. We call these operations tropical addition and tropical multiplication and denote these operations by $\oplus$ and $\odot$ respectively. We note that 0 plays the role of the multiplicative identity and $-\infty$ plays the role of the additive identity. The semiring was coined a 'tropical' semiring by a French mathematician by the name of Dominique Perrin [14] in honour of a Brazilian mathematician named Simon Imre who wrote the foundational material on the max-plus semiring [22]. Hence the word 'tropical' here has no other meaning but to represent the French view of Brazil. This semiring and its associated linear spaces have been studied extensively in the context of computer science (see [3] and references therein).

To define tropical matrix operations over the max-plus semiring, one simply replaces operations of addition and multiplication in normal matrix operations with their tropical equivalents. Hence we define the tropical analogues of matrix addition and multiplication by $\oplus$ and $\odot$ respectively by the equations

$$(A \oplus B)_{ij} := \max(a_{ij}, b_{ij})$$
$$(A \odot B)_{ij} := \max_k(a_{ik} + b_{kj})$$

where $A = (a_{ij})$ and $B = (b_{ij})$. For any $s \in S$ we define a scalar product by the equation

$$(s \odot A)_{ij} = s + a_{ij}.$$

This provides a tropical matrix setting for our Birkhoff theory.

In addition to constructing linear spaces over S, we wish to have a notion in which things converge. We endow S with the metric

$$d(x, y) = |\mathrm{e}^x - \mathrm{e}^y| \tag{2}$$

where we define $\mathrm{e}^{-\infty}$ to be 0. The advantage of this metric is that when we restrict our attention to the topology on U, we have the same topology as the induced topology on $\mathbb{R}$. One important difference is that we consider sequences tending to $-\infty$ to be convergent.

Given a subtraction-free q-difference equation, it is a routine procedure to obtain a corresponding ultradiscrete equation over S. The procedure is called ultradiscretization. This process was originally used to draw a link between the box-and-ball system and the discrete KdV equation [23]. Given a rational expression in a set of positive real variables, say $f(a_1, a_2, \ldots, a_n)$, we introduce a set of ultradiscrete variables, say A_i, related to the original variables via the relation $a_i = \mathrm{e}^{A_i/\epsilon}$. The ultradiscrete analogue of the rational expression is given by the limit

$$F(A_1, \ldots, A_n) = \lim_{\epsilon \to 0} \epsilon \log f(a_1, \ldots, a_n). \tag{3}$$

The process can be alternatively stated by giving the following correspondences between expressions over $\mathbb{R}^+$ and S. Given $a = \mathrm{e}^{A/\epsilon}$ and $b = \mathrm{e}^{B/\epsilon}$, the ultradiscretization procedure is as follows:

$$a + b \to \max(A, B) \tag{4a}$$

$$ab \to A + B \tag{4b}$$

$$a/b \to A - B. \tag{4c}$$

By replacing all operations specified on the left-hand side of (4) with the corresponding operation on the right-hand side, one derives the required ultradiscrete expression. For example, given the q-difference equation

$$w(qt)w(t/q) = \frac{at + w^2}{1 + atw^2},$$

it is easily verifiable that the ultradiscretization of this equation is (1). Ultradiscrete analogues of all the Painlevé equations have been given along with many of the properties that these equations possess [15]. It is the integrability of these equations that we wish to provide evidence for.

The first endeavour one may pursue is to consider what the ultradiscrete analogue of singularity confinement would be. However, the guiding principle for the singularity confinement of discrete equations is somewhat lost for equations over semirings. The 'singularities' no-longer manifest themselves over S in the same way since there is no way to obtain an ∞ using operations of max and +. In fact many methods one wishes to apply over a field are lost when one's domain is the semiring. The information in the discrete equation is lost, imposing a series of constraints and problems :

- Any rational expression you wish to find the ultradiscrete analogue of is required to be subtraction free.
- As expressions over S, one loses information. For example, as an expression over S, $\max(0, x, 2x) = \max(0, 2x)$.
- The semiring S has no subtraction. For example, the linear equation $\max(0, X) = -1$ has no solution.

Although these constraints are unavoidable when working over S, we may be able to exploit a higher space to derive results for systems over S. These very results may not be obtainable when restricting one's attention to operations over S alone. This concept is more natural than it sounds. For example, there exist linear difference equations over $\mathbb{N}$ whose solutions may not be expressed using operations over $\mathbb{N}$ alone, but are easily expressed over $\mathbb{Q}$ or $\mathbb{C}$. We intend to do the same. In order to do this, we review some well-established concepts in algebra.

Definition 1. *A non-Archimedean valuation ring is a ring, R, with a valuation, $v : R \to \mathbb{R} \cup \{-\infty\}$, such that*

(1) $v(x) = -\infty$ *if and only if* $x = 0$.

(2) $v(xy) = v(x) + v(y)$.
(3) $v(x + y) \leqslant \max(v(x), v(y))$.

We note that the valuation v is 'almost' a homomorphism from R into S. There are many examples of non-Archimedean valuation fields in the literature:

- $\mathbb{C}[t]$ = the polynomials in $\mathbb{C}$ over t with valuation
$$v : a_0 + a_1 t + \cdots + a_n t^n \to n$$
where $a_n \neq 0$.
- $\mathbb{Q}$ under the valuation
$$v : \frac{a}{b} p^n \to -n$$
where $a, b \nmid p$. This is the p-adic valuation.
- $K = \overline{\mathbb{C}(x)}$ with valuation v which is the unique extension of the order function that brings a function to the negative of the order of its pole or 0 to the algebraic closure. As a function $v : K \to \mathbb{Q}$.

It is interesting to note that varieties in the tropical sense can be expressed as the topological closures of images of varieties under the valuation over K [18]. Disjoint from the theory of [18] comes a sense of convergence from a metric induced by the valuation:

$$d(x, y) = \mathrm{e}^{v(x-y)}. \tag{5}$$

In this sense, the tropical geometric framework combined with the analytic framework fuels the approach we have adopted for examining associated linear problems. We may develop a systematic way of tackling the theory of associated linear problems over S using this framework. However, the above examples of fields are too simple, or insufficient in some manner, for our purposes. The first two fields are discrete valuation fields, and the last assumes that the additively closed subset of the reals we used to form S is $\mathbb{Q}$. We introduce the ring that has suited our purposes.

Let Φ be the ring of all formal $\mathbb{Z}$ linear combinations of elements of U. We denote elements of Φ by

$$x = \sum n_i (x_i)$$

where $n_i \in \mathbb{Z}$ and $x_i \in U$ for all i. We may consider U as an additive group, in which case Φ is a group ring. The ring possesses operations of + and $\times$. If $x = \sum n_i(x_i)$ and $y = \sum m_i(y_i)$ the operations are defined by the equations

$$x + y = \sum_i n_i(x_i) + \sum_i m_i(y_i) \tag{6a}$$

$$x \times y = \sum_{i,j} n_i m_j (x_i + y_j). \tag{6b}$$

Let Ω be the field of fractions of Φ. We will denote elements of Ω by $\frac{x}{y}$ or by the ratio of formal linear combinations that x and y represent. We endow Ω with a valuation, $P : \Omega \to S$, defined by

$$P : \frac{x}{y} \to \max_i (x_i) - \max_i (y_j). \tag{7}$$

We also define $P(0) = -\infty$. The metric defined in terms of the valuation gives us a sense in which sequences in Ω converge. We identify a sub-semiring given by

$$\Omega_0 = \left\{ \frac{\sum n_i x_i}{\sum m_i y_i} \text{ such that } m_i, n_i \in \mathbb{N} \right\}.$$

We note that $P|_{\Omega_0} : \Omega \to S$ is a homomorphism of semirings.

Remark. As algebraic objects, the field Ω is isomorphic to the inversible max-plus algebra [11]. As an ideological discrepancy, we do not insist that equations over Ω are of importance in themselves. We intend to provide evidence that lifting max-plus equations to the level of a field may be useful in determining properties of systems over S.

Given an element over S, we identify the standard lift to be the mapping $s \to 1(s)$ (i.e. as a $\mathbb{Z}$-linear combination, it is a single element whose $\mathbb{Z}$ component is 1 and real component is S). For any equation, matrix or scalar we may identify an equivalent equation in Ω_0 recoverable through P. As we will see later, it may be preferable to not use a standard lift. Given $x \in S$, $P(1(x)) = x$. If $P(y) \leqslant x$ then $P(1(x) + y) = x$. We call the mapping $1(x) \to 1(x) + y$ a projection preserving transformation.

2. Systems of linear difference equations over the max-plus semiring

We draw upon some classical results before delving into the crux of our theory. We start by considering a system of linear difference equations of the form

$$Y(qx) = A(x)Y(x) \tag{8}$$

where A is some rational matrix function in x and $q \in \mathbb{C}$ is fixed. If $|q| > 1$ then we have the following symbolic solutions at 0 and ∞ as infinite products:

$$Y_0 = A(x/q)A(x/q^2)A(x/q^3)\ldots \tag{9a}$$

$$Y_\infty = A^{-1}(x)A^{-1}(qx)A^{-1}(q^2x)\ldots. \tag{9b}$$

The theory of Birkhoff concerns (amongst other things) when such solutions define convergent sequences. It is well known that such a problem can be transformed to the case in which A is polynomial. We may then assume that A is of the form

$$A(x) = A_0 + A_1 x + \cdots + A_n x^n.$$

One classical result we attribute to Carmichael [2] is as follows:

Theorem 1. *If A_0 and A_n are semisimple with eigenvalues $\lambda_0, \ldots, \lambda_m$ and $\mu_0, \ldots, \mu_m$ such that $\lambda_i/\lambda_j \notin q^{\mathbb{Z}}$ and $\mu_i/\mu_j \notin q^{\mathbb{Z}}$ then (9a) and (9b) define holomorphic functions with possible poles at $zq^{\mathbb{Z}}$ where* $\det A(z) = 0$. *Furthermore we have the following forms,*

$$Y_0(x) = \hat{Y}_0 x^{D_0} \tag{10a}$$

$$Y_\infty(x) = \hat{Y}_\infty x^{D_n} q^{\frac{n}{2}u(u-1)} \tag{10b}$$

where $D_0 = \operatorname{diag}(\log_q(\lambda_i))$, $D_n = \operatorname{diag}(\log_q(\mu_i))$ *and* $u = \log_q x$.

If such solutions exist, then we may define the connection matrix associated with (8) to be

$$M(x) = (Y_\infty(x))^{-1} Y_0(x) = \ldots A(qx)A(x)A(x/q)\ldots \tag{11}$$

which is obviously q-periodic in x. If we introduce a variable, t, and allow the coefficients A_i to be functions of t, we may write

$$A(x) = A(x, t)$$
$$Y(x) = Y(x, t)$$
$$M(x) = M(x, t).$$

If the system is such that $M(x, t) = M(x, qt)$, we call the transformation $Y(x, t) \to Y(x, qt)$ a connection preserving transformation. A necessary condition for this to occur is that Y must satisfy the linear system given by the equation

$$Y(x, qt) = B(x, t)Y(x, t) \tag{12}$$

where $B = Y_\infty(x, qt)(Y_\infty(x, t))^{-1} = Y_0(x, qt)(Y_0(x, t))^{-1}$. It is important to note that the matrix B is rational in x. This extension imposes a compatibility when attempting to calculate $Y(qx, qt)$. This forces the condition

$$A(x, qt)B(x, t) = B(qx, t)A(x, t).$$

It was surprising, albeit amazing to discover that q-P_{VI} and the q-Garnier equation came as necessary compatibility conditions for connection preserving transformations [8, 20]. This provides further evidence that connection preserving deformations are the discrete analogue of isomondromic deformations of linear systems [5].

We will show how one may transfer such results to the max-plus semiring. In the max-plus semiring setting, we consider systems of linear difference equations given by

$$Y(X + Q) = A(X) \odot Y(X) \tag{13}$$

where A is tropically rational in X. We may transform this system analogously, reducing the equation to one in which A is polynomial in X using the max-plus analogue of θ functions, thus we may assume that A is of the form

$$A(X) = A_0 \oplus A_1 \odot X \oplus \cdots \oplus A_n \odot nX.$$

Unlike linear systems of q-difference equations, we do not have the luxury of being able to invert matrices in general. Necessary constructs to define the conditions for the existence of a connection matrix (such as eigenvalues) have no analogous manifestation over S. The only solvable case at a glance seems to be the set of tropically linear scalar equations; these are solved with not much difficulty at all. One is required to find solutions to the scalar case to transform A from a rational matrix to a polynomial matrix in X over S. For general systems of linear difference equations, we need to rely on a different set of tools. Since invertible matrices are a rarity in the max-plus setting, we are not able to analogously define solutions at both 0 and ∞, but we are able to define

$$Y_0(X) = A(X - Q) \odot A(X - 2Q) \odot A(X - 3Q) \ldots \tag{14a}$$
$$(Y_\infty(X))^{-1} = \ldots A(X + 2Q) \odot A(X + Q) \odot A(X) \tag{14b}$$

under conditions in which these converge. This would be sufficient to define a connection matrix over S given by

$$M(X) = (Y_\infty(X))^{-1} \odot Y_0(X). \tag{15}$$

We will be concerned when such a matrix can be defined. We are required to lift the problem to Ω_0 in order to make some headway on the problem.

It is now convenient to consider the analogous problem over Ω. This is given by the system of linear difference equations given by

$$\mathcal{Y}(1(X+Q)) = \mathcal{A}(1(X))\mathcal{Y}(1(X)) \tag{16}$$

where $\mathcal{A}$ is of the form

$$\mathcal{A}(1(X)) = \mathcal{A}_0 + \mathcal{A}_1 1(X) + \cdots + \mathcal{A}_n 1(X)^n. \tag{17}$$

As a matter of notation, it is convenient to write $F(1(X))$ to mean the function of the real variable X taking values over Ω. Furthermore, by letting the $\mathcal{A}_i$ be matrices over Ω_0 such that $P(\mathcal{A}_i) = A_i$, then this is an analogous system to the system over S via P. The canonical choice of matrices would be given by the standard lift. We are interested in the following symbolic solutions:

$$\mathcal{Y}_0(1(X)) = \mathcal{A}(1(X-Q))\mathcal{A}(1(X-2Q))\mathcal{A}(1(X-3Q))\ldots \tag{18a}$$

$$\mathcal{Y}_\infty(1(X)) = \mathcal{A}(1(X))^{-1}\mathcal{A}(1(X+Q))^{-1}\mathcal{A}(1(X+2Q))^{-1}\ldots. \tag{18b}$$

This infinite series can be expressed as the limit of a series of matrices over Ω, which we require to be convergent over Ω. We will extend Ω by taking the closure under the metric. We now are in a position to state the main theorem as shown in [12].

Theorem 2. *Suppose $\mathcal{A}_0$ and $\mathcal{A}_n$ are semisimple over Ω with eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_m$ respectively. If $|P(\lambda_i) - P(\lambda_j)| < Q$ and $|P(\mu_i) - P(\mu_j)| < Q$ for all i, j then the symbolic solutions (18a) and (18b) define convergent functions in some open set of Ω. Furthermore, have the forms*

$$\mathcal{Y}_0(X) = \hat{\mathcal{Y}}_0(X)D_0^{\frac{X}{Q}} \tag{19a}$$

$$\mathcal{Y}_{-\infty}(X) = \hat{\mathcal{Y}}_\infty(X)D_n^{\frac{X}{Q}}1\left(\frac{nX(X-Q)}{2Q}\right) \tag{19b}$$

where $D_0 = \mathrm{diag}(\lambda_i)$ and $D_n = \mathrm{diag}(\mu_i)$.

If (18a) and (18b) exist, then we may define the connection matrix over Ω given by

$$\mathcal{M}(1(X)) = (\mathcal{Y}_\infty(1(X)))^{-1}\mathcal{Y}_0(1(X))$$

which is expected to be pseudo-constant under the evolution $1(X) \to 1(X)1(Q) = 1(X+Q)$.

Semisimplicity over Ω is a rather strict condition due to the fact that the field is not algebraically closed. If we were to take the algebraic closure of the field, or even adjoin appropriate eigenvalues as a field extension, it would be unclear as to how to define P on the field extensions. This would give us some ambiguity in how we state of the conditions in the theorem. Although this is a rather large constraint for systems over Ω, there are an infinite number of systems over Ω that map to the same system over S. This allows for a relaxation of the conditions for the existence of a connection matrix over S significantly.

Corollary 1. *Let $\mathcal{A}_i$ be a set of matrices over Ω_0 such that $P(\mathcal{A}_i) = A_i$ such that the system defined by equations (16), (17) possesses solutions (18a) and (18b). Under these conditions (13) possesses a solution given by (14a) and the formal inverse of a solution given by (14b).*

The crux of this relaxation lies in the conditions for there to exist a projection preserving transformation that brings a matrix that is not semisimple to one that is semisimple. There are a plethora of examples in which one may apply projection preserving transformations to the standard lift of the problem so that the resulting linear problem over Ω satisfies the conditions of the main theorem. This tool is useful for deriving extensions to the above corollary that are based on a handful of simple inequalities that may be explicitly stated over S. One may also use such theory to easily find powers of matrices over S.

We are now in a position to define the connection matrix over S to be the product

$$M(X) = (Y_\infty(X))^{-1} \odot Y_{-\infty}(X).$$

We are now in a position to introduce a connection variable T. We have over Ω that

$$\begin{aligned}
\mathcal{A}(1(X)) &= \mathcal{A}(1(X), 1(T)) \\
\mathcal{A}_i &= \mathcal{A}_i(1(T)) \\
\mathcal{Y}(1(X)) &= \mathcal{Y}(1(X), 1(T)) \\
\mathcal{M}(1(X)) &= \mathcal{M}(1(X), 1(T))
\end{aligned}$$

so over S we have

$$\begin{aligned}
A(X) &= A(X, T) \\
A_i &= A_i(T) \\
Y(X) &= Y(X, T) \\
M(X) &= M(X, T).
\end{aligned}$$

Just as in the q-difference case, if we constrain the system by the relation $\mathcal{M}(1(X), 1(T)) = \mathcal{M}(1(X), 1(T + Q))$ over Ω, it is a necessary condition that $\mathcal{Y}$ satisfies another linear equation

$$\mathcal{Y}(1(X), 1(T + Q)) = \mathcal{B}(1(X), 1(T))\mathcal{Y}(1(X), 1(T)).$$

If in addition we may transform the problem (through some projection preserving transformation of $\mathcal{A}$ or otherwise) so that $\mathcal{B}$ is a matrix over Ω_0 then we may consistently extend (13). This extension over S may be characterized by the equation

$$Y(X, T + Q) = B(X, T) \odot Y(X, T)$$

where $B(X, T) = P(\mathcal{B}(1(X), 1(T))$. We then have the following compatibility condition over S:

$$A(X, T + Q) \odot B(X, T) = B(X + Q, T) \odot A(X, T). \tag{20}$$

This is the equivalent derivation of a Lax pair over a max-plus semiring. We consider any compatibility condition arising in this manner a necessary condition for a connection preserving transformation. We consider this evidence of integrability for any system equivalent to this compatibility condition.

3. Example

This example was introduced by the author *et al* [9] as a ultradiscretized version of q-$\mathbb{P}_{III}$. In relation to (1), we study the linear system (13) in which the matrix A is given by

$$A(x, t) = A_0 \oplus A_1 \odot X \oplus A_2 \odot 2X$$

where the coefficient matrices, A_i, are given by

$$A_0 = \begin{pmatrix} -\infty & \frac{A}{2} - \frac{3Q}{2} + W \\ \frac{A}{2} - \frac{3Q}{2} - W & -\infty \end{pmatrix}$$
$$A_1 = \begin{pmatrix} \overline{W} - W - Q & -\infty \\ -\infty & W - \overline{W} - Q) \end{pmatrix}$$
$$\oplus \begin{pmatrix} A - Q + T + W + \overline{W} & -\infty \\ -\infty & A - Q + T - W - \overline{W} \end{pmatrix}$$
$$A_2 = \begin{pmatrix} -\infty & \frac{A}{2} - \frac{Q}{2} + T - W \\ \frac{A}{2} - \frac{Q}{2} + T + W & -\infty \end{pmatrix}.$$

The lifted matrix required over Ω_0 is given by the standard lift $(\mathcal{A} \to (1(a_{ij})))$. The eigenvalues of $\mathcal{A}_0$ and $\mathcal{A}_2$ are given by the solutions to the characteristic equations

$$\lambda^2 - 1(A - 3Q) = 0$$
$$\mu^2 - 1(A - Q + 2T) = 0.$$

The eigenvalues are $\lambda_1 = 1\left(\frac{A}{2} - \frac{3Q}{2}\right), \lambda_2 = -1\left(\frac{A}{2} - \frac{3Q}{2}\right), \mu_1 = 1\left(\frac{A-Q}{2} + T\right)$ and $\mu_2 = -1\left(\frac{A-Q}{2} + T\right)$. We have the equalities $P(\lambda_1) = P(\lambda_2)$ and $P(\mu_1) = P(\mu_2)$; this tells us the conditions for the corollary are met. This implies A possesses a connection matrix and that we may derive the appropriate extension of the linear system by defining the B matrix over S. We let

$$B(X, T) = B_0 \oplus (B_1 \odot X)$$

where the coefficient matrices, B_i, are given by

$$B_0 = \begin{pmatrix} \overline{W} - W & -\infty \\ -\infty & 0 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} -\infty & \frac{A}{2} + \frac{Q}{2} + T - W \\ \frac{A}{2} + \frac{Q}{2} + T + \overline{W} & -\infty \end{pmatrix}.$$

This Lax pair is equivalent to that found in [9] in which one necessary condition imposed by the compatibility is (1). If one naively takes (20) to be the compatibility condition then the left- and right-hand side have $\overline{\overline{W}}$ inside the max statements, making it unclear how the compatibility condition describes the evolution of W. The easiest way to see the compatibility condition is through the fact that A contains a factor of B on the right, meaning that we may obtain the matrix $\tilde{A}$ given by

$$\tilde{A}(X, T) = \begin{pmatrix} X + \overline{W} - W & A \\ -Q & X + W - \overline{W} \end{pmatrix}$$

that acts on our linear system in a manner such that $Y(X + Q, T) = \tilde{A}(X, T) \odot Y(X, T + Q)$. The equivalent compatibility condition is given by

$$\begin{aligned} Y(X + Q, T + Q) &= \tilde{A}(X, T + Q) \odot Y(X, T + 2Q) \\ &= \tilde{A}(X, T + Q) \odot B(X, T + Q) \odot Y(X, T + Q) \\ Y(X + Q, T + Q) &= B(X + Q, T) \odot Y(X + Q, T) \\ &= B(X + Q, T) \odot \tilde{A}(X, T) \odot Y(X, T + Q) \end{aligned}$$

which implies

$$\tilde{A}(X, T + Q) \odot B(X, T + Q) = B(X + Q, T) \odot \tilde{A}(X, T).$$

Looking at respective parts, the off-diagonal entries give identities while the diagonal entries give the conditions

$$\overline{\overline{W}} - \overline{W} + \max(0, A + T + Q + 2\overline{W}) = -W - \overline{W} + \max(A + T + Q, 2\overline{W})$$
$$-\overline{W} - \overline{\overline{W}} + \max(A + T + Q, 2\overline{W}) = -\overline{W} + W + \max(0, A + T + Q + 2\overline{W}).$$

The compatibility conditions in (21) are both equivalent to (1). Hence we consider (1) as having been derived as a compatibility condition resulting from a connection matrix preserving deformation of linear systems.

4. Conclusion

There are many directions one can pursue with this theory. One intriguing possibility is the formulation of Galois theory similar to the theory of [21] for linear systems over S. It would also be interesting to see how one may develop this theory (and the q-difference theory) in a direction analogous to inverse scattering and inverse monodromy [4]. This would reinforce this notion of integrability through a connection matrix approach. Another possibility is the prospect of an application of tropical algebraic geometry to the classification of these systems. Many more practical applications pertain to the exploration of how to use such liftings to find special solutions and symmetries of equations over S.

Acknowledgment

This research was supported in part by the Australian Research Council grant #DP0559019.

References

[1] Birkhoff G D 1911 General theory of linear difference equations *Trans. Am. Math. Soc.* **12** 243–84
[2] Carmichael R D 1912 The general theory of linear q-difference equations *Am. J. Math.* **34** 147–68
[3] Cohen G, Richet-Gebert S and Quadrat J-P 1999 Max-plus algebra and system theory: where we are and where to go now *Ann. Rev. Control* **23** 207–19
[4] Fokas A S, Muğan and Ablowitz M J 1988 A method of linearization for Painlevé equations: Painlevé IV, V *Physica* D **30** 247–83
[5] Fuchs R 1907 Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen *Math. Ann.* **63** 301–21
[6] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67** 1825–8
[7] Grammaticos B and Ramani A 1996 Painlevé equations and cellular automata: symmetries and integrability of difference equations *London Math. Soc. Lecture Note Ser.* **255** 325–33
[8] Jimbo M and Sakai H 1996 A q-analogue of the sixth Painlevé equation *Lett. Math. Phys.* **38** 145–54
[9] Joshi N, Nijhoff F and Ormerod C M 2004 Lax pairs for ultra-discrete Painlevé cellular automata *J. Phys. A: Math. Gen.* **37** L559–L565
[10] Joshi N and Lafortune S 2005 How to detect integrability in cellular automata *J. Phys. A: Math. Gen.* **38** L499–L504
[11] Ochiai T and Nacher J 2005 Inversible max-plus algebra and Integrable systems *J. Math. Phys.* **46** 063507
[12] Ormerod C Ultradiscrete connection matrices over a tropical semiring, at press
[13] Papageorgiou V G, Nijhoff F W, Grammaticos B and Ramani A 1991 Isomonodromic deformation problems for discrete analogues of Painlevé equations *Phys. Lett.* A **164** 57–64
[14] Pin J 1998 Tropical semi-rings: idempotency *Publ. Newton Inst.* **11** 50–69
[15] Ramani A, Takahashi D, Grammaticos B and Ohta Y The ultimate discretisation of the Painlevé equations *Physica* D **114** 185–96
[16] Ramani A, Grammaticos B and Hietarinta J 1991 Discrete versions of the Painlevé equations *Phys. Rev. Lett.* **67** 1829–32

[17] Quispel G R W, Capel H W and Scully J 2001 Piecewise-linear soliton equations and piecewise-linear integrable maps *J. Phys. A: Math. Gen.* **34** 2491–503

[18] Richter-Gebert J, Sturmfels B and Theobald T 2003 First steps in tropical geometry *Preprint* math.AG/0306366

[19] Sakai H 2001 Rational surfaces associated with affine root systems and geometry of the Painlevé equations *Commun. Math. Phys.* **220** 165–229

[20] Sakai H 2005 A q-analogue of the Garnier system *Funkcial. Ekvac* **48** 273–97

[21] Sauloy J 2003 Galois thoery of Fuchsian q-difference equations *Ann. Sci. École Norm. Sup.* (4) **36** 925–68

[22] Simon I 1978 Limited subsets of a free monoid *19th Annual Symp. on Foundations of Computer Science (Ann Arbor, MI) (Long Beach, CA: IEEE)* pp 143–50

[23] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 From soliton equations to integrable cellular automata through a limiting procedure *Phys. Rev. Lett.* **76** 3247–50

[24] Viallet C-M and Bellon M P 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37